

Random and Correlated Phases of Primordial Gravitational Waves

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The phases of primordial gravity waves is analysed in detail within a quantum mechanical context following the formalism developed by Grishchuk and Sidorov. It is found that for physically relevant wavelengths both the phase of each individual mode and the phase *difference* between modes are randomly distributed. The phase *sum* between modes with oppositely directed wave-vectors, however, is not random and takes on a definite value with no rms fluctuation. The conventional point of view that primordial gravity waves appear after inflation as a classical, random stochastic background is also addressed.

It is well known that gravitational waves (GW) can be produced in inflationary cosmologies [1]. They arise from quantum fluctuations in the gravitational field on a curved de Sitter space-time. The generated perturbations are then often treated as a Gaussian noise with randomly distributed phases. During de Sitter expansion, it is believed that these fluctuations are then stretched “out” of the horizon where they “freeze” and remain constant until much later when they re-enter the horizon and appear as classical GW. These GW then form a classical stochastic background which fills the universe. It has, however, also been argued that the quantum-mechanical generation of perturbations, in contrast to a classical, stochastic perturbation spectrum, actually possesses very specific statistical properties. Specifically, it has been argued that the phases of all modes of perturbations are essentially constant and fixed. Moreover, strong quantum correlations cause classical standing GW to form leading to important cosmological consequences [4].

In this paper we shall present a detailed analysis of the statistical properties of the phases of relic GW from a quantum mechanical viewpoint. As the generation of these GW are fundamentally quantum mechanical in nature we shall thus be able to comment on the view that these GW subsequently form a classical, stochastic background. Our results show that the phase of each individual mode is random. More importantly, in all but one case, there is also the absence of any correlation between the phases of two different modes. Namely, both the phase sum and phase difference between any two arbitrary modes vanish. There exists, however, one exceptional case, that of two modes with oppositely directed wave-vectors, for which the *phase sum* of the two modes are highly-correlated. This is a manifestation of phase locking between modes which also occurs in quantum optics [3]. The phase difference between these two modes are still completely uncorrelated and random, however. In many instances, therefore, the relic GW may be *approximated* as a “classical” stochastic background. Nevertheless, the phase-sum locking asserts that the background is fundamentally *quantum mechanical* in nature. Based on these results, we shall comment on the usual classical treatment of relic GW.

In our analysis we shall follow the framework developed by Grishchuk and Sidorov [4]. But instead of using ‘standing wave’ operators which decompose the two-mode squeeze operator into the product of two single mode operators, we shall stay with the original

‘traveling wave’ operators where the physics is more straightforwardly seen. We warn the reader, however, that while the number operator for each mode is invariant under either choice, the phase operator is not and our definition of the phase operator differs from that given in [4]

The quantized graviton field operator h_{ij} may be written as

$$h_{ij}(\eta, \vec{x}) = C \sum_{\vec{q}} \sum_{s=1}^2 p_{ij}^s(\vec{q}) \left[a_{\vec{q}}^s(\eta) e^{i\vec{q} \cdot \vec{x}} + a_{\vec{q}}^{s\dagger}(\eta) e^{-i\vec{q} \cdot \vec{x}} \right], \quad (1)$$

where C is an overall constant whose value is not important for our purposes, $p_{ij}^s(\vec{q})$ are the polarization tensors and s labels the two polarization states of the wave. $a_{\vec{q}}^s(\eta)$ and $a_{\vec{q}}^{s\dagger}(\eta)$ are raising and lowering operators in the Heisenberg representation and with their evolution governed by the hamiltonian

$$H = \sum_{\vec{q}} \sum_{s=1}^2 \left\{ q a_{\vec{q}}^{s\dagger}(\eta) a_{\vec{q}}^s(\eta) + q a_{-\vec{q}}^{s\dagger}(\eta) a_{-\vec{q}}^s(\eta) + 2\sigma(\eta) \left[a_{\vec{q}}^{s\dagger}(\eta) a_{-\vec{q}}^{s\dagger}(\eta) - a_{\vec{q}}^s(\eta) a_{-\vec{q}}^s(\eta) \right] \right\} \quad (2)$$

where $\sigma(\eta) = iR'/2R$. Here R is the cosmic scale factor and the prime denotes derivative with respect to η .

Notice that the graviton field is coupled to the background metric, and thus to the expansion of the universe, through a conformal time varying *quadratic* interacting hamiltonian. This hamiltonian may be diagonalized and the Heisenberg evolution equations

$$i da_{\vec{q}}^s/d\eta = [a_{\vec{q}}^s, H], \quad i da_{\vec{q}}^{s\dagger}/d\eta = -[a_{\vec{q}}^{s\dagger}, H], \quad (3)$$

solved exactly using a Bogolubov transformation. In fact, Grishchuk and Sidorov make the time dependent Bogolubov transformation: $a_{\vec{q}}^s(\eta) \rightarrow u_q(\eta) a_{\vec{q}}^s(\eta_0) + v_q(\eta) a_{-\vec{q}}^{s\dagger}(\eta_0)$, $a_{\vec{q}}^{s\dagger}(\eta) \rightarrow \bar{u}_q(\eta) a_{\vec{q}}^{s\dagger}(\eta_0) + \bar{v}_q(\eta) a_{-\vec{q}}^s(\eta_0)$ where η_0 is some initial conformal time. Requiring this to be a canonical transformation restricts

$$|u_q(\eta)|^2 - |v_q(\eta)|^2 = 1, \quad (4)$$

for all η . Then, noting that in the presence of inflation $R'/R \rightarrow 0$ as $\eta \rightarrow -\infty$, they choose as the initial conditions for the Bogolubov coefficients to be $u_q(\eta_0) = 1$ and $v_q(\eta_0) = 0$ where η_0 is some initial time in the far past.

While this is certainly a valid approximation, one eventually finds that it is more useful to write

$$u_q(\eta) = e^{i\epsilon_q} \cosh r_q, \quad v_q(\eta) = e^{-i(\epsilon_q - 2\phi_q)} \sinh r_q, \quad (5)$$

which explicitly satisfies eq. (4). The functions $r_q(\eta)$, $\epsilon_q(\eta)$ and $\phi_q(\eta)$ are called the squeeze parameter, rotation angle, and squeeze angle, respectively. If one then uses Grishchuk's initial conditions, one finds that $r_q(\eta_0) = 0$ and $\epsilon_q(\eta_0) = 0$. The initial condition for ϕ_q is *undetermineable*, however, and it is this phase which will play a crucial role in determining the phase of the GW. A more careful analysis must there be done to establish the correct initial condition for it.

To do so, let us begin by defining

$$\begin{aligned} \alpha_{\vec{q}}^s(\eta) &= u_q(\eta) a_{\vec{q}}^s(\eta_0) + v_q(\eta) a_{-\vec{q}}^{s\dagger}(\eta_0), \\ \alpha_{\vec{q}}^{s\dagger}(\eta) &= \bar{u}_q(\eta) a_{\vec{q}}^{s\dagger}(\eta_0) + \bar{v}_q(\eta) a_{-\vec{q}}^s(\eta_0), \end{aligned} \quad (6)$$

where $a_{\vec{q}}^s(\eta_0)$ and $a_{\vec{q}}^{s\dagger}(\eta_0)$ are evaluated at some initial time η_0 . Then from the Heisenberg evolution equations, u_q and v_q evolve as

$$\begin{aligned} \frac{du_q}{d\eta} &= -iq u_q + \frac{R'}{R} \bar{v}_q, \\ \frac{dv_q}{d\eta} &= -iq v_q + \frac{R'}{R} \bar{u}_q. \end{aligned} \quad (7)$$

The initial conditions for u_q and v_q are given at η_0 and are fixed by requiring $\alpha_{\vec{q}}^s(\eta_0)$ and $\alpha_{\vec{q}}^{s\dagger}(\eta_0)$ to diagonalize H *at this time*:

$$H(\eta_0) = \sum_{\vec{q}} \sum_{s=1}^2 q e(q) \left[\alpha_{\vec{q}}^{s\dagger}(\eta_0) \alpha_{\vec{q}}^s(\eta_0) + \alpha_{-\vec{q}}^{s\dagger}(\eta_0) \alpha_{-\vec{q}}^s(\eta_0) \right], \quad (8)$$

since it is only in this way that a ground state for the system can be defined. From the standard Bogolubov analysis, we find that

$$u_q(\eta_0) = i e^{i\theta} \sqrt{\frac{1+e}{2e}}, \quad v_q(\eta_0) = -e^{i\theta} \sqrt{\frac{1-e}{2e}}, \quad (9)$$

and

$$e(q) = \sqrt{1 - \left(\frac{R'(\eta_0)}{qR(\eta_0)} \right)^2}. \quad (10)$$

Here θ is an arbitrary phase for each mode q . We fix it by requiring that $u_q(\eta_0) \rightarrow 1$ if the interaction term is turned off: $|\sigma| \rightarrow 0$ so that $\theta = -\pi/2$ for all q . Notice also that because

$e(q)$ must be real number, the initial time η_0 must be chosen such that $2|\sigma(\eta_0)| < q$ for all \vec{q} . Usually η_0 is chosen to be at such an early time that $|\sigma(\eta_0)| \approx 0$ and this is not a problem. From Eq. (9), $\cosh r_q(\eta_0) = \sqrt{(1+e)/2e}$, and $\epsilon_q(\eta_0) = 0$. These agree with Grishchuk's initial conditions in the limit $e(q) \rightarrow 1$. The initial condition for $\phi_q(\eta_0) = -\pi/4$ can now be determined, however.

As was first noticed by [4], the above system is equivalent to what are called squeezed states in quantum optics. Namely, one notices that the transformed operators are related to the original operators through a unitary transformation:

$$\begin{aligned}\alpha_{\vec{q}}^s(\eta) &= \mathcal{R}_{\vec{q}}^s(\eta) \mathcal{S}_{\vec{q}}^s(\eta) a_{\vec{q}}^s(\eta_0) \mathcal{S}_{\vec{q}}^{s\dagger}(\eta) \mathcal{R}_{\vec{q}}^{s\dagger}(\eta), \\ \alpha_{\vec{q}}^{s\dagger}(\eta) &= \mathcal{R}_{\vec{q}}^s(\eta) \mathcal{S}_{\vec{q}}^s(\eta) a_{\vec{q}}^{s\dagger}(\eta_0) \mathcal{S}_{\vec{q}}^{s\dagger}(\eta) \mathcal{R}_{\vec{q}}^{s\dagger}(\eta),\end{aligned}\tag{11}$$

where

$$\mathcal{R}_{\vec{q}}^s(\eta) = \exp \left\{ -i\epsilon_q(\eta) \left(a_{\vec{q}}^{s\dagger}(\eta_0) a_{\vec{q}}^s(\eta_0) + a_{-\vec{q}}^{s\dagger}(\eta_0) a_{-\vec{q}}^s(\eta_0) \right) \right\}, \tag{12}$$

is the two mode rotation operator while

$$\mathcal{S}_{\vec{q}}^s(\eta) = \exp \left\{ r_q(\eta) \left(e^{-2i\phi_q(\eta)} a_{\vec{q}}^s(\eta_0) a_{-\vec{q}}^s(\eta_0) - e^{2i\phi_q(\eta)} a_{\vec{q}}^{s\dagger}(\eta_0) a_{-\vec{q}}^{s\dagger}(\eta_0) \right) \right\} \tag{13}$$

is the two mode squeeze operator.

It is currently held that quantum fluctuations in the gravitational field will be amplified as the universe undergoes a fast expansion. (See, for example [5] for a complete description of particle creation in the universe). This can be seen explicitly by looking at

$$\langle 0 | n_{\vec{q}}^s | 0 \rangle = \langle 0 | \alpha_{\vec{q}}^{s\dagger}(\eta) \alpha_{\vec{q}}^s(\eta) | 0 \rangle = |v_q(\eta)|^2 = \sinh^2 r_q(\eta), \tag{14}$$

for any mode \vec{q} . It is then argued that at the present time these primordial, quantum fluctuations will appear as classical GW with a classical stochastic distribution of random phases. This, however, has never been explicitly established.

It is known [6] that the number operator and the phase operator

$$\exp\{i\varphi_k^s(\eta)\} \equiv \left(I + n_k^s(\eta) \right)^{-1/2} \alpha_k^s(\eta), \quad \exp\{-i\varphi_k^s(\eta)\} \equiv \alpha_k^{s\dagger}(\eta) \left(I + n_k^s(\eta) \right)^{-1/2}, \tag{15}$$

for the graviton field do not commute. The phase itself is not an hermitian operator, however, and suffers from a problem with multiplicity. We shall instead have to work with the operators

$$\begin{aligned}\cos \varphi_k^s(\eta) &\equiv \frac{1}{2} \left(\exp\{i\varphi_k^s(\eta)\} + \exp\{-i\varphi_k^s(\eta)\} \right) , \\ \sin \varphi_k^s(\eta) &= \frac{1}{2i} \left(\exp\{i\varphi_k^s(\eta)\} - \exp\{-i\varphi_k^s(\eta)\} \right) ,\end{aligned}\tag{16}$$

which are hermitian, and thus physical observables. One then finds that

$$[n_k^s, \cos \varphi_k^s] = -i \sin \varphi_k^s, \quad [n_k^s, \sin \varphi_k^s] = i \cos \varphi_k^s, \tag{17}$$

with the corresponding uncertainty relations

$$\Delta n_k^s e \cos \varphi_k^s \geq \frac{1}{2} |\langle \sin \varphi_k^s \rangle|, \quad \Delta n_k^s e \sin \varphi_k^s \geq \frac{1}{2} |\langle \cos \varphi_k^s \rangle|, \tag{18}$$

where $\Delta A \equiv \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ is the rms fluctuation of the operator. The classical limit for massless particles, therefore, involves not only looking at the intensity of the radiation, but also at its phase.

Using eq. (5), the rms fluctuation in the number operator for each mode is

$$\Delta n_k^s = |u_k| |v_k| = \sqrt{2} \cosh r_k(\eta) \sinh r_k(\eta) \tag{19}$$

so that $\Delta n_k^s / \langle n_k^s \rangle = \sqrt{2} / \tanh r_k$. Now, it is usually stated [4] that although $\langle 0 | h_{ij} | 0 \rangle = 0$, because $\langle 0 | n_k^s | 0 \rangle \neq 0$, for \vec{k} mode with a large squeeze parameter one can still view each mode of the GW as a classical wave with amplitude $A_{\vec{k}} = \langle 0 | n_k^s | 0 \rangle^{1/2}$. One can certainly make this interpretation, but it does give the impression that the classical wave will have a well defined, definite amplitude. From eq. (19), we see that this need not be the case. In fact, in inflationary cosmologies [4], modes just entering the horizon at the present day would have a $r_k \sim 120$ giving $\Delta n_k^s \sim \langle n_k^s \rangle$. The fluctuation of the amplitude of this classical wave will be on the order of its amplitude itself. In this case, we *cannot* characterize the resultant ‘classical’ wave as having any definite amplitude.

Let us now consider the phases of these GW. We begin by calculating the average phase of each mode

$$\langle 0 | \cos \varphi_q^s(\eta) | 0 \rangle = \langle 0 | \mathcal{R}_q^s(\eta) \mathcal{S}_q^s(\eta) \cos \varphi_q^s(\eta_0) \mathcal{S}_q^{s\dagger}(\eta) \mathcal{R}_q^{s\dagger}(\eta) | 0 \rangle, \tag{20}$$

where we have used eq. (11). Since $\mathcal{R}_q^{s\dagger} | 0 \rangle = | 0 \rangle$, and by using the following factorization [7],

$$\begin{aligned}
\mathcal{S}_q^s &= \frac{1}{\cosh r_q} \exp \left\{ -a_q^{s\dagger}(\eta_0) a_{-q}^{s\dagger}(\eta_0) e^{2i\phi_q} \tanh r_q \right\} \\
&\exp \left\{ - \left[a_q^{s\dagger}(\eta_0) a_q^s(\eta_0) + a_{-q}^{s\dagger}(\eta_0) a_{-q}^s(\eta_0) \right] \ln \cosh r_q \right\} \\
&\exp \left\{ a_q^s(\eta_0) a_{-q}^s(\eta_0) e^{-2i\phi_q} \tanh r_q \right\}.
\end{aligned} \tag{21}$$

Then

$$\mathcal{S}_q^{s\dagger}|0\rangle = \frac{1}{\cosh r_q} \sum_{n=0}^{\infty} (e^{2i\phi_q} \tanh r_q)^n |n, n\rangle, \tag{22}$$

so that $\langle 0 | \cos \varphi_q^s(\eta) | 0 \rangle = 0$. Similarly, $\langle 0 | \sin \varphi_q^s(\eta) | 0 \rangle = 0$. The average phase of any one mode vanishes, as expected for randomly distributed phases. This, however, does not tell us whether or not this random distribution is “classical”. To do so, we calculate the rms fluctuation in the phase

$$(\Delta \cos \varphi_q^s)^2 = \langle 0 | (\cos \varphi_q^s)^2 | 0 \rangle = \frac{1}{2} - \frac{1}{4 \cosh^2 r_q}, \tag{23}$$

with $\langle 0 | (\cos \varphi_q^s)^2 | 0 \rangle = \langle 0 | (\sin \varphi_q^s)^2 | 0 \rangle$.

In eq. (23) we can see a deviation from the classical stochastic behavior. Suppose that the phase of this mode is classical in nature and can be described by a random, stochastic behavior. Denoting the phase of this mode by θ_q , then because everything are c-numbers, $\cos^2 \theta_q + \sin^2 \theta_q = 1$. If the phase of this mode is random, we would expect the stochastic average $\langle \cos^2 \theta_q \rangle_{sto} = \langle \sin^2 \theta_q \rangle_{sto}$. Consequently, for a classical, random stochastic distribution of phase, one would expect $\langle \cos^2 \theta_q \rangle_{sto} = 1/2$.

Since, however, $(\cos \varphi_q^s)^2$ and $(\sin \varphi_q^s)^2$ are *operators*, their sum need not add up to unity. As such, for a random distribution their expectation value (average) need not be 1/2, as it is for the classical stochastic distribution. In fact, any deviation from 1/2 is a sign of the quantum nature of the mode. As we can see from eq. (23), the quantum nature of the mode is always present, but becomes progressively smaller for large r_p . Modes just entering the horizon at the present day have a $r_q \sim 120$ and for these modes $\langle 0 | (\cos \varphi_q^s)^2 | 0 \rangle \approx 1/2$. They are therefore essentially classical in nature and using a random, stochastic distribution to describe their phase would be correct. Notice, however, that for $r_q \sim 1$, deviation from the classical behavior becomes pronounced and these modes are essentially quantum mechanical in nature.

Of course, the absolute phase of any one mode is irrelevant. What is more interest is the relative phases between modes. Given the phase operator for any one mode, we follow [6] and define the phase sum and difference operators between any two modes as

$$\begin{aligned}\sin(\varphi_{\vec{p}}^s \pm \varphi_{\vec{q}}^t) &\equiv \sin \varphi_{\vec{p}}^s \cos \varphi_{\vec{q}}^t \pm \cos \varphi_{\vec{p}}^s \sin \varphi_{\vec{q}}^t, \\ \cos(\varphi_{\vec{p}}^s \pm \varphi_{\vec{q}}^t) &\equiv \cos \varphi_{\vec{p}}^s \cos \varphi_{\vec{q}}^t \mp \sin \varphi_{\vec{p}}^s \sin \varphi_{\vec{q}}^t.\end{aligned}\tag{24}$$

Notice that in the limit $\vec{p} \rightarrow \vec{q}$, $t = s$ the sine difference operator *does not vanish* since $[\cos \varphi_{\vec{q}}^t, \sin \varphi_{\vec{q}}^t] \neq 0$. This once again underscores the fact that we are dealing with operators and not functions.

It is then straightforward to show that for all t , s , and $\vec{p} \neq -\vec{q}$,

$$\langle 0 | \sin(\varphi_{\vec{p}}^s \pm \varphi_{\vec{q}}^t) | 0 \rangle = 0, \quad \langle 0 | \cos(\varphi_{\vec{p}}^s \pm \varphi_{\vec{q}}^t) | 0 \rangle = 0, \tag{25}$$

as expected since the average phase of each mode vanishes. What is of more interest is the expectation value of the squares of these operators

$$\begin{aligned}\langle 0 | (\sin(\varphi_{\vec{p}}^s - \varphi_{\vec{q}}^t))^2 | 0 \rangle &= \langle 0 | (\cos(\varphi_{\vec{p}}^s - \varphi_{\vec{q}}^t))^2 | 0 \rangle = \frac{1}{4}(\tanh^2 r_p + \tanh^2 r_q), \\ \langle 0 | (\sin(\varphi_{\vec{p}}^s + \varphi_{\vec{q}}^t))^2 | 0 \rangle &= \langle 0 | (\cos(\varphi_{\vec{p}}^s + \varphi_{\vec{q}}^t))^2 | 0 \rangle = \frac{1}{4}(1 + \tanh^2 r_p \tanh^2 r_q).\end{aligned}\tag{26}$$

Once again in the limit of large r_p and r_q , these results go over to what one expects for a classical stochastic distribution of the phases. Consequently, we find that as long as $\vec{p} \neq -\vec{q}$ the phase difference and sum between modes are completely random and in the limit of large r_p and r_q , can be accurately approximated as classical stochastic distribution of the phases.

When $t = s$ and $\vec{p} = -\vec{q}$ the situation changes quite dramatically, however. First, we find that

$$\langle 0 | \sin(\varphi_{\vec{p}}^s - \varphi_{-\vec{p}}^s) | 0 \rangle = 0, \quad \langle 0 | \cos(\varphi_{\vec{p}}^s - \varphi_{-\vec{p}}^s) | 0 \rangle = 0, \tag{27}$$

as expected. But now

$$\begin{aligned}\langle 0 | \sin(\varphi_{\vec{p}}^s + \varphi_{-\vec{p}}^s) | 0 \rangle &= \sin 2\phi_p \tanh r_p, \\ \langle 0 | \cos(\varphi_{\vec{p}}^s + \varphi_{-\vec{p}}^s) | 0 \rangle &= \cos 2\phi_p \tanh r_p.\end{aligned}\tag{28}$$

Moreover,

$$\langle 0 | (\sin(\varphi_{\vec{p}}^s - \varphi_{-\vec{p}}^s))^2 | 0 \rangle = \langle 0 | (\cos(\varphi_{\vec{p}}^s - \varphi_{-\vec{p}}^s))^2 | 0 \rangle = \frac{1}{2} \tanh^2 r_p, \quad (29)$$

while

$$\begin{aligned} \langle 0 | (\sin(\varphi_{\vec{p}}^s + \varphi_{-\vec{p}}^s))^2 | 0 \rangle &= \frac{1}{4} \left(1 - \tanh^2 r_p + 4 \tanh^2 r_p \sin^2(2\phi_p) \right), \\ \langle 0 | (\cos(\varphi_{\vec{p}}^s + \varphi_{-\vec{p}}^s))^2 | 0 \rangle &= \frac{1}{4} \left(1 - \tanh^2 r_p + 4 \tanh^2 r_p \cos^2(2\phi_p) \right). \end{aligned} \quad (30)$$

Once again, in the limit of large r_p , we find that $(\Delta \sin(\varphi_{\vec{p}}^s - \varphi_{-\vec{p}}^s))^2 = 1/2$, meaning that phase difference between the two modes is completely random. The fluctuation in the phase sum $\Delta \sin(\varphi_{\vec{p}}^s + \varphi_{-\vec{p}}^s) = 0$, however, and the two modes have a definite *phase sum*. This is a manifestation of two mode phase locking first calculated by [3] using distribution function methods.

We thus see that although the phase differences between modes are uncorrelated and completely random, the phase sum between the \vec{p} and $-\vec{p}$ modes with the same polarization is highly correlated with an average value of $2\phi_p$ and essentially no fluctuation whatsoever at large r_p . This correlation was also founded by Grishchuk and Sidorov and was interpreted by them as the formation of a classical standing wave. Note, however, that this phenomenon is inherently quantum mechanical in nature and it is impossible to explain both eqs. (29) and (30) using classical stochastic arguments for the following reasons.

Suppose that we wish to explain the results of the phase sum and difference analysis for $\vec{p} = -\vec{q}$ in the large r_p limit using classical stochastic arguments. Then eq. (28) implies that the phase distribution of both the \vec{p} and $-\vec{p}$ modes are peaked at ϕ_p . That there is a width to this distribution can be seen in eq. (29) and in fact we see that there must be a randomly distributed background noise below ϕ_p . If, however, this background noise is present, then one would not expect eq. (30) to hold, since although we would expect the average phase sum to be $2\phi_p$, we still expect the noise to be present and would not expect the rms fluctuation in the phase sum to vanish. Classically, it should be $1/2$ once more. Consequently, the results of the phase sum and difference analysis cannot be explained using classical methods.

The phase locking of the \vec{p} and $-\vec{p}$ modes also suggests the following method to measure the thermal history of the universe. Notice that for large r_p , which correspond to most

physically relevant modes, the phase sum of the two modes is $2\phi_p$ with essential no fluctuation whatsoever. From eq. (7), we see that the value of ϕ_p depends explicitly on the scale factor R which is in turn determined by the thermal history of the universe. Consequently, we propose to fix a specific direction along the celestial sphere and measure the phase sum of primordial GW with various momentum. This will determine each ϕ_p . From eq. (14), r_p can be inferred by measuring the amplitude of the wave, although from eq. (19) the fluctuation in this amplitude is expected to be large. Through these two measurements the thermal history along that direction can be extracted. Next, fix the magnitude of \vec{p} and measure the phase sum $= 2\phi_p$ for a series of angles on the celestial sphere. If the universe was truly isotropic throughout its history, then ϕ_p should be independent of angle. If not, then variation in ϕ_p will be a measure of the anisotropy of the universe. In either case, a complete determination of ϕ_p and r_p will be useful for probing the very early universe.

How these experiments would be done is not as yet known. Not only hasn't any GW been detected yet (not to mention a primordial one), this experiment would also require measuring the quantum mechanical phase sum of two modes. It has only been very recently that the phase operator for optical waves has been measured experimentally [8], and no experimental measurement of phase locking has yet been found even for optical waves. Consequently, we would expect such experiments, if they can be done at all, can only be accomplished in the far, far future. Nevertheless, they have the potential to provide a very clean and direct measurement of the history of the universe.

To conclude, we have in this paper completed a detailed quantum mechanical analysis of the properties of GW arising from fluctuations in the de Sitter vacuum of the universe. We find that the present day characterization of these waves as classical GW with a classical, stochastic distribution in the phase to be somewhat naive. First, while one can identify the rms fluctuation of each mode of the graviton field h_{ij} as a classical amplitude, the fluctuation of this 'amplitude' is very large for large r_p , the most physically relevant ones. Next, although the average phase of each mode separately is randomly distributed and can be well approximated as a classical stochastic distribution, there are non-classical, strong correlations between the phases of different \vec{p} . There is a definite value of the phase sum of the \vec{p} and $-\vec{p}$ modes, although the phase difference of these modes, as it is between any two

modes, is completely random and stochastically distributed. This has been interpreted by Grishchuk and Sidorov as being due to the formation of standing waves in the universe.

We now can understand the limitations of the usual classical treatment of the relic GW. In this treatment, the time evolution of the amplitude of classical GW generated from inflation is obtained from a two-point statistical average of a stochastic ensemble of classical gravitational fields which is then matched to the quantum mechanical two-point function of the gravitational field [2]. The subsequent evolution of the classical waves is then described by the classical wave propagation in the expanding universe. This constitutes the so-called stochastic classical background of relic GW with randomly distributed phases. Intuitively, we say that during inflationary expansion fluctuations in the gravitational field will be red-shifted out of the horizon, after which they freeze and remain at a constant amplitude. Much later when they re-enter the horizon during either the radiation- or matter-dominated era, they will appear as classical oscillating GW.

There are, however, two different phases which must be considered if one wishes to view the system in this way. The first is the *temporal* phase of oscillation of waves. This is precisely fixed in inflationary cosmologies. Once a frozen mode starts to oscillating as a classical wave at the time it crosses the horizon and re-enters the universe, the temporal phase is completely determined by the classical equation of motion. However, inflation does not predict the *location* of the nodes of these frozen modes since quantum fluctuations assign equal probability to modes which differ only by a spatial translation. Herein lies the other phase which must be considered: that due to the spatial part of the GW. It is this phase which is completely random. Consequently, in this classical treatment the overall phase of GW are randomly distributed.

Based on our results of the statistical properties of the phases of GW, the above classical description has to be amended. Since the phases of different modes are almost uncorrelated, in many aspects the relic GW can be well as a stochastic classical background radiation. Phase-sum locking, however, is still present and is a strong indication of its inherent quantum character of these GW. Moreover, this separation of the phase of each mode into spatial and temporal parts is completely artificial and arises only from the desire to use a classical description and interpretation of the quantum two-point function. From the experience of

calculating the vacuum expectation values of the phase operators, we can hardly separate the temporal phase from the spatial phase. This is not surprising, though, since there is only one single overall phase in the quantum mechanical treatment. Consequently, although the classical approach advocated in [2] is quite physical and is, as we have seen, a good approximation of an inherently quantum mechanical phenomenon in many instances, it can not fully describe the properties of the relic GW. Our results suggest that one should instead use the quantum mechanical approach to deal with these GW.

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REFERENCES

- ¹ A. A. Starobinskii, *Pis'ma Zh. Eksp. Teor. Fiz.* **30**, 719 (1979) [JETP Lett. **30**, 682 (1979)].
- ² L. F. Abbott and M. B. Wise, *Nucl. Phys.* **B244**, 541 (1984); L. F. Abbott and D. D. Harari, *Nucl. Phys.* **B264**, 487 (1986).
- ³ S. M. Barnett and D. T. Pegg, *Phys. Rev.* **A42**, 6713 (1990).
- ⁴ L. P. Grishchuk and Y. V. Sidorov, *Phys. Rev.* **D42**, 3413 (1990); see also L. P. Grishchuk, *Class. Quantum Grav.* **10**, 2449 (1993).
- ⁵ N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Chapters 3, 5, (Cambridge University Press, New York, 1982).
- ⁶ P. Carruthers and M. M. Nieto, *Phys. Rev. Lett.* **14**, 387 (1965); see also P. Carruthers and M. M. Nieto, *Rev. Mod. Phys.* **40**, 411 (1968).
- ⁷ C. M. Caves and B. L. Shumaker, *Phys. Rev.* **A31**, 3068 (1985); B. L. Shumaker and C. M. Caves, *Phys. Rev.* **A31**, 3093 (1985).
- ⁸ J. W. Noh, A. Fougères, and L. Mandel, *Physica Script.* **T48**, 29 (1993); D. T. Smithey, M. Beck, J. Cooper, and M. G. Raymer, *Physica Script.* **T48**, 35 (1993); A. Fougères, J. W. Noh, T. P. Grayson, and L. Mandel, *Phys. Rev.* **A49**, 530 (1994).